

# The Functional Synthesis of Linear Plots<sup>1</sup>

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In practical engineering or experimental work one often encounters a function  $F$  of many variables,  $F(x$ 's,  $y$ 's,  $z$ 's), represented only by the families of curves obtained by plotting  $F$  against each of the  $x$ 's on Cartesian graph paper, against each of the  $y$ 's on semi-log paper, and against each of the  $z$ 's on double-log paper. It frequently happens that these curves are all approximately straight lines over a limited range of interest. On the assumption that they are all true straight lines, the present note shows how to synthesize all the graphical representations, for any number of parameters, into the most general formula possible, expressing  $F$  as the product of a multilinear function of the  $x$ 's and the exponential of a constant-free multilinear function of the  $y$ 's and of the log  $z$ 's, the coefficients in both multilinear functions being independent of the  $x$ 's,  $y$ 's, and  $z$ 's.

## 1. Introduction

In an investigation by one of the authors on assemblages of elastic shells, the various results for certain components of stress and displacement exhibited approximately linear behavior over the limited ranges of interest of the relevant parameters, when plotted on Cartesian graph paper, semi-log paper, or double-log paper. These results, derived from various numerical calculations and corroboratory experiments, are functions of many parameters, such as the various geometrical ratios defining shell shapes, the shell thickness, Poisson's ratio, the number of coupled shells, and other significant quantities. Rather than to retain these extensive results in the form of cumbersome families of graphs, it was desirable and useful to combine them in such a way as to obtain a single explicit formula for each dependent quantity, in terms of the above-mentioned independent parameters.

The problem of combining such results is, of course, not peculiar to investigations in elasticity but often arises in experimental or engineering work of any nature. One can very easily construct simple functions which will exhibit some of these "linear" properties, but the most general answer to the inverse problem is less obvious, especially when the number of parameters is large. Then an unsystematic attempt to effect such a synthesis may prove infeasible or incomplete, and furnish no assurance that one has indeed constructed the most general function with these properties. Our present note, therefore, formulates the general problem for any number of variables in a precise manner and derives its *most general* solution. For the simple case of only three parameters, the solution is illustrated by a few examples.

## 2. The General Problem

In the following developments we shall assume that all the pertinent quantities have been grouped into independent dimensionless combinations, so as to take advantage of whatever information is provided by the Buckingham Pi theorem.<sup>3</sup> Let us then consider a dimensionless function  $F$  of the dimensionless independent variables  $x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_p; z_1, z_2, \dots, z_q$ ; and  $u_1, u_2, \dots, u_r$ , such that a straight line results when we plot  $F$  against any  $x$  on ordinary Cartesian graph paper, against any  $y$  on semi-log paper, or against any  $z$  on double-log paper. Let the  $u$ 's denote all other dimensionless variables on which  $F$  may depend, but for which no such "linear" property exists. It is understood, of course, that in practice such linearity may hold only approximately, and only for a certain bounded range of the variable used as abscissa, and only for certain bounded ranges of the remaining variables, which appear as parameters. With no loss of generality, however, we relax these restrictions, assuming that each plot is a straight line for all values of abscissa and parameters.

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<sup>3</sup> P. W. Bridgman, Dimensional analysis, p. 36 (Yale Univ. Press, New Haven, Conn., 1931).

We put

$$\ln z_1 = y_{p+1}, \quad \ln z_2 = y_{p+2}, \quad \dots, \quad \ln z_q = y_n, \quad (n = p + q); \quad (1)$$

then  $F(x_1, \dots, x_m, y_1, \dots, y_n, u)$  is a function such that the plot of  $F$  against any  $x$  or of  $\ln F$  against any  $y$  is a straight line on Cartesian graph paper. Here  $u$  denotes the set  $u_1, u_2, \dots, u_r$ . Then

$$\ln F = A_k y_k + B_k \quad (k = 1, 2, \dots, n) \quad (2)$$

where  $A_k$  and  $B_k$  are functions of  $x_1, \dots, x_m, y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n$ , and of the  $u$ 's. We then ask: if  $\ln F$  is linear in each  $y$  when the other  $y$ 's are all held constant, what is the most general form for  $\ln F$ , as a function of the  $y$ 's, that will represent such a property?

To answer this preliminary question, we first recall the definition of a *multilinear* function  $G$ , of several variables  $t_1, t_2, \dots, t_s$ , as a function which is the sum of a constant and a linear combination of all the products of the  $t$ 's taken one at a time, two at a time,  $\dots$ ,  $s$  at a time, without repetition. For example, if  $s = 3$ ,

$$G(t_1, t_2, t_3) = a_0 + a_1 t_1 + a_2 t_2 + a_3 t_3 + b_1 t_2 t_3 + b_2 t_3 t_1 + b_3 t_1 t_2 + c t_1 t_2 t_3. \quad (3)$$

Any such multilinear function satisfies the differential equations

$$\partial^2 G / \partial t_k^2 = 0 \quad (k = 1, 2, \dots, s). \quad (4)$$

(When the constant term vanishes, we term  $G$  a constant-free multilinear function.) It is then easily shown by induction that  $G(t_1, t_2, \dots, t_s)$  is the most general function of  $t_1, t_2, \dots, t_s$  which is linear in each  $t$ .

## 2.1. The Synthesis for the $y$ 's Alone

On applying these considerations to (2), we find that

$$\ln F = g(x_1, x_2, \dots, x_m, u) + N^{(x, u)}(y_1, y_2, \dots, y_n), \quad (5)$$

where  $N^{(x, u)}$  denotes a general constant-free multilinear function of the  $y$ 's, with coefficients which are, a priori, functions of  $x_1, \dots, x_m, u$ . On placing

$$\exp g = f(x_1, x_2, \dots, x_m, u), \quad (6)$$

we obtain

$$F = f(x_1, x_2, \dots, x_m, u) \exp N^{(x, u)}, \quad (7)$$

which gives the synthesis of the linearities of  $\ln F$  versus the  $y$ 's.

## 2.2. The Complete Synthesis

Since  $F$  is linear in each  $x$ , it follows from the property of multilinear functions that

$$F = P^{(y, u)}(x_1, x_2, \dots, x_m), \quad (8)$$

where  $P^{(y, u)}$  is a general multilinear function of the  $x$ 's, the coefficients being functions of the  $y$ 's and the  $u$ 's. Then

$$F = f(x_1, x_2, \dots, x_m, u) \exp N^{(x, u)} = P^{(y, u)}(x_1, x_2, \dots, x_m), \quad (9)$$

where by (4)

$$\partial^2 N / \partial y_k^2 = 0 \quad (k = 1, 2, \dots, n) \quad (10)$$

$$\partial^2 P / \partial x_j^2 = 0 \quad (j = 1, 2, \dots, m). \quad (11)$$

We next show that  $f$  is a multilinear function of the  $x$ 's. To do so, put every  $y$  equal to zero in (9). Since  $N$  contains no constant term, it then vanishes, so that (9) becomes

$$f(x_1, x_2, \dots, x_m, u) = M^{(u)}(x_1, x_2, \dots, x_m). \quad (12)$$

Here  $M^{(u)}$  is simply the expression for  $P^{(y, u)}$  with each coefficient evaluated at each  $y=0$ , so that it is a multilinear function of the  $x$ 's with coefficients depending only upon the  $u$ 's.

We now show that the coefficients in  $N^{(x, u)}$  are independent of the  $x$ 's, as follows. Insert (12) into (7), so that

$$F = M^{(u)} \exp N^{(x, u)} \quad (13)$$

and require that (13) satisfy (11). It follows that

$$\frac{\partial^2 M}{\partial x_j^2} + M \frac{\partial^2 N}{\partial x_j^2} + 2 \frac{\partial M}{\partial x_j} \frac{\partial N}{\partial x_j} + M \left( \frac{\partial N}{\partial x_j} \right)^2 = 0 \quad (14)$$

where we have omitted the superscripts for convenience. Since  $M$  is multilinear in the  $x$ 's, it follows from (4) that

$$\partial^2 M / \partial x_j^2 = 0, \quad (15)$$

so that

$$M \frac{\partial^2 N}{\partial x_j^2} + 2 \frac{\partial M}{\partial x_j} \frac{\partial N}{\partial x_j} + M \left( \frac{\partial N}{\partial x_j} \right)^2 = 0. \quad (16)$$

If we now differentiate (16) twice with respect to  $y_k$ , we find with use of (10) that

$$2M \left( \frac{\partial^2 N}{\partial x_j \partial y_k} \right)^2 = 0, \quad (17)$$

whence

$$\frac{\partial}{\partial y_k} \left( \frac{\partial N}{\partial x_j} \right) = 0 \quad \begin{cases} j=1, 2, \dots, m, \\ k=1, 2, \dots, n. \end{cases} \quad (18)$$

Equation (18) means that  $\partial N / \partial x_j$  can be a function only of the  $x$ 's and the  $u$ 's. But  $N$  and  $\partial N / \partial x_j$  are constant-free multilinear functions of the  $y$ 's, with coefficients that are, a priori, functions of the  $x$ 's and the  $u$ 's. These results are compatible if and only if

$$\partial N / \partial x_j = 0 \quad (j=1, 2, \dots, m) \quad (19)$$

so that each coefficient in the constant-free multilinear function  $N$  must be independent of the  $x$ 's.

We may thus rewrite (13) as

$$F = M^{(u)}(x_1, x_2, \dots, x_m) \exp N^{(u)}(y_1, y_2, \dots, y_n). \quad (20)$$

When we return to the original formulation of the problem in terms of the  $x$ 's,  $y$ 's, and  $z$ 's, it follows that the *most general* functional form for  $F$  is given by

$$F = M^{(u)}(x_1, x_2, \dots, x_m) \exp N^{(u)}(y_1, y_2, \dots, y_p, \ln z_1, \ln z_2, \dots, \ln z_q), \quad (21)$$

where  $M^{(u)}$  is a general multilinear function of the  $x$ 's and  $N^{(u)}$  a general constant-free multilinear function of the  $y$ 's and  $\ln z$ 's, the coefficients in both being functions only of the  $u$ 's.

### 3. Some Elementary Examples

As short illustrations of the general result (21), we append a few cases where  $F$  has linear plots against only three variables. For each we list the specific form that (21) assumes and the slopes and intercepts on the appropriate plots. By comparing the behavior of the slopes.

and intercepts in the various experimentally given families of curves that define the function  $F$  with these listed formulas, one can readily determine which coefficients vanish, if any, and thus obtain a specific formula for  $F$  in any actual case. Here "ln" denotes a natural logarithm and "log" a common logarithm. For an  $x$  or a  $y$  the intercept is taken at zero, while for a  $z$  it is taken at  $z=1$ . For the logarithmic plots the slopes and intercepts are those of  $\log F$ . In the following formulas, it is understood that the constants may be functions of the  $u$ 's.

(a) **x, y<sub>1</sub>, y<sub>2</sub>**

$$F = (k_1x + k_2) \exp(a_1y_1 + a_2y_2 + by_1y_2)$$

*Cartesian* plot versus  $x$ :

$$\text{slope} \quad S = k_1 \exp(a_1y_1 + a_2y_2 + by_1y_2)$$

$$\text{intercept} \quad I = k_2 \exp(a_1y_1 + a_2y_2 + by_1y_2)$$

*Semi-log* plot versus  $y_1$ :

$$\text{slope} \quad S = 0.4343(a_1 + by_2)$$

$$\text{intercept} \quad I = \log(k_1x + k_2) + 0.4343a_2y_2$$

(b) **x, y, z**

$$F = (k_1x + k_2)e^{a_1yz^{a_2} + by}$$

*Cartesian* plot versus  $x$ :

$$\text{slope} \quad S = k_1e^{a_1yz^{a_2} + by}$$

$$\text{intercept} \quad I = k_2e^{a_1yz^{a_2} + by}$$

*Semi-log* plot versus  $y$ :

$$\text{slope} \quad S = 0.4343a_1 + b \log z$$

$$\text{intercept} \quad I = \log(k_1x + k_2) + a_2 \log z$$

*Double-log* plot versus  $z$ :

$$\text{slope} \quad S = a_2 + by$$

$$\text{intercept} \quad I = \log(k_1x + k_2) + 0.4343a_1y$$

(c) **y<sub>1</sub>, y<sub>2</sub>, z**

$$F = ke^{a_1y_1 + a_2y_2 + b_3y_1y_2}z^{a_3 + b_1y_1 + b_2y_2 + cy_1y_2}$$

*Semi-log* plot versus  $y_1$ :

$$\text{slope} \quad S = 0.4343(a_1 + b_3y_2) + (b_1 + cy_2)\log z$$

$$\text{intercept} \quad I = \log k + 0.4343a_2y_2 + (a_3 + b_2y_2)\log z$$

*Double-log* plot versus  $z$ :

$$\text{slope} \quad S = a_3 + b_1y_1 + b_2y_2 + cy_1y_2$$

$$\text{intercept} \quad I = \log k + 0.4343(a_1y_1 + a_2y_2 + b_3y_1y_2)$$

(d) **y, z<sub>1</sub>, z<sub>2</sub>**

$$F = ke^{a_1yz_1^{a_2} + b_1yz_2^{a_3} + b_2yH}$$

where

$$H = z_1^{(b_3+cy)\ln z_2} = z_2^{(b_3+cy)\ln z_1}$$

*Semi-log* plot versus  $y$ :

$$\text{slope} \quad S = 0.4343a_1 + b_1 \log z_1 + b_2 \log z_2 + 2.303c \log z_1 \log z_2$$

$$\text{intercept} \quad I = \log k + a_2 \log z_1 + a_3 \log z_2 + 2.303b_3 \log z_1 \log z_2$$

*Double-log* plot versus  $z_1$ :

$$\text{slope} \quad S = a_2 + b_1 y + 2.303(b_3 + cy) \log z_2$$

$$\text{intercept} \quad I = \log k + 0.4343a_1 y + (a_3 + b_2 y) \log z_2$$

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